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# About a Brooks-type theorem for improper colouring\*

Ricardo Corrêa<sup>†</sup>      Frédéric Havet<sup>‡</sup>      Jean-Sébastien Sereni<sup>§</sup>

## Abstract

A graph is  $k$ -improperly  $\ell$ -colourable if its vertices can be partitioned into  $\ell$  parts such that each part induces a subgraph of maximum degree at most  $k$ . A result of Lovász states that for any graph  $G$ , such a partition exists if  $\ell \geq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$ . When  $k = 0$ , this bound can be reduced by Brooks' Theorem, unless  $G$  is complete or an odd cycle. We study the following question, which can be seen as a generalisation of the celebrated Brooks' Theorem to improper colouring: does there exist a polynomial-time algorithm that decides whether a graph  $G$  of maximum degree  $\Delta$  has  $k$ -improper chromatic number at most  $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$ ? We show that the answer is no, unless  $\mathcal{P} = \mathcal{NP}$ , when  $\Delta = \ell(k+1)$ ,  $k \geq 1$ , and  $\ell + \sqrt{\ell} \leq 2k + 3$ . We also show that, if  $G$  is planar,  $k = 1$  or  $k = 2$ ,  $\Delta = 2k + 2$ , and  $\ell = 2$ , then the answer is still no, unless  $\mathcal{P} = \mathcal{NP}$ . These results answer some questions of Cowen *et al.* [Journal of Graph Theory 24(3):205-219, 1997].

## Introduction

An  $\ell$ -colouring of a graph  $G = (V, E)$  is a mapping  $c : V \rightarrow \{1, 2, \dots, \ell\}$ . For any vertex  $v \in V$ , the *impropriety of  $v$  under  $c$*  is

$$\text{im}_c(v) := |\{u \in V : uv \in E \text{ and } c(u) = c(v)\}|.$$

A colouring is  $k$ -improper provided that the impropriety of every vertex is at most  $k$ . A 0-improper colouring is *proper*. A graph is  $k$ -improperly  $\ell$ -colourable if it admits a  $k$ -improper  $\ell$ -colouring. The  $k$ -improper chromatic number is

$$c_k(G) := \min\{\ell : G \text{ is } k\text{-improperly } \ell\text{-colourable}\}.$$

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In particular,  $c_o(G)$  is the chromatic number  $\chi(G)$  of the graph  $G$ . Since the early nineties, a lot of work has been devoted to various aspects of improper colourings, both from a purely theoretical point of view [7, 8, 14, 20, 21] and in relation with frequency assignment issues [12, 13, 16]. Let us note that improper colourings are also called in the literature *defective colourings*.

For all integers  $k$  and  $\ell$ , let  $k$ -IMP  $\ell$ -COL be the following problem:

INSTANCE: a graph  $G$ .

QUESTION: is  $G$   $k$ -improperly  $\ell$ -colourable?

Cowen *et al.* [8] showed that the problem  $k$ -IMP  $\ell$ -COL is  $\mathcal{NP}$ -complete for all pairs  $(k, \ell)$  of integers with  $k \geq 1$  and  $\ell \geq 2$ . When  $\ell \geq 3$ , this is not very surprising since it is already hard to determine whether a given graph is properly 3-colourable. On the contrary, determining if a graph is 2-colourable, i.e. bipartite, can be done in polynomial-time, whereas it is  $\mathcal{NP}$ -complete to know if it is  $k$ -improper 2-colourable as soon as  $k > 0$ .

Of even more interest is the question of complexity of  $k$ -IMP  $\ell$ -COL when restricted to graphs with maximum degree  $(k+1)\ell$ . Indeed, Lovász [17] proved that, for any graph  $G$ , it holds that  $c_k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$ , where  $\Delta(G)$  is the maximum degree of  $G$ . When  $k = 0$ , this is the usual bound  $\chi(G) \leq \Delta(G) + 1$ . Brooks' Theorem [6] states that this upper bound can be decreased by one, provided that  $G$  is neither complete nor an odd cycle, which can be checked in polynomial-time.

Extensions of Brooks' Theorem have also been considered. A well-known conjecture of Borodin and Kostochka [5] states that every graph of maximum degree  $\Delta \geq 9$  and chromatic number at least  $\Delta$  has a  $\Delta$ -clique. Reed [19] proved that this is true when  $\Delta$  is sufficiently large, thus settling a conjecture of Beutelspacher and Herring [4]. Further information about this problem can be found in the monograph of Jensen and Toft [15, Problem 4.8]. Generalisation of this problem has also been studied by Farzad, Molloy, and Reed [11] and Molloy and Reed [18]. In particular, it is proved [18] that determining whether a graph with large constant maximum degree  $\Delta$  is  $(\Delta - k)$ -colourable can be done in linear time if  $(k+1)(k+2) \leq \Delta$ . This threshold is optimal by a result of Emden-Weinert, Hougardy, and Kreuter [10], since they proved that for any two constants  $\Delta$  and  $k \leq \Delta - 3$  such that  $(k+1)(k+2) > \Delta$ , determining whether a graph of maximum degree  $\Delta$  is  $(\Delta - k)$ -colourable is  $\mathcal{NP}$ -complete.

It is natural to ask whether analogous results can be found for improper colouring. This first problem to grapple with is the existence, or not, of a Brooks-like theorem for improper colouring: does there exist a polynomial-time algorithm that decides whether a graph  $G$  of maximum degree  $\Delta$  has  $k$ -improper chromatic number at most  $\left\lceil \frac{\Delta+1}{k+1} \right\rceil - 1$ ? Proving that  $k$ -IMP  $\ell$ -COL is  $\mathcal{NP}$ -complete when restricted to graphs with maximum degree  $(k+1)\ell$  would provide a negative answer to this question unless  $\mathcal{P} = \mathcal{NP}$ . Cowen *et al.* [8] proved that  $k$ -IMP 2-COL is  $\mathcal{NP}$ -complete for the class of graphs with maximum degree  $2(k+1)$ , and asked what happens when  $\ell \geq 3$ . In this paper, we prove that  $k$ -IMP  $\ell$ -COL restricted to graphs with maximum degree  $(k+1)\ell$  is  $\mathcal{NP}$ -complete for all integers  $k \geq 1$  and  $\ell \in \{3, \dots, s\}$ , where  $s$  is the biggest integer such that  $s + \sqrt{s} \leq 2k + 3$  (Theorem 2). An intriguing question that remains unanswered is the complexity of this

problem for larger values of  $\ell$ .

**Problem 1.** What is the complexity of  $k$ -IMP  $\ell$ -COL restricted to graphs with maximum degree  $(k+1)\ell$  when  $\ell + \sqrt{\ell} > 2k+3$ ?

We conjecture that is always  $\mathcal{NP}$ -complete. As an evidence, we prove the  $\mathcal{NP}$ -completeness when  $k=1$  and  $\ell=4$  (Theorem 6).

In view of these negative results, one may ask what happens for planar graphs. It is known that every planar graph is 4-colourable [1, 3, 2], 2-improperly 3-colourable [9, 21], and Cowen *et al.* [8] proved that it is  $\mathcal{NP}$ -complete to know whether a planar graph is 1-improperly 3-colourable, but without any restriction on the maximum degree.

Cowen *et al.* [8] also proved that  $k$ -IMP 2-COL is  $\mathcal{NP}$ -complete for planar graphs, again without any restriction on the degree. In particular, they asked if 1-IMP 2-COL is still  $\mathcal{NP}$ -complete for planar graphs with maximum degree 4 — they could prove it only for maximum degree 5. In more general terms, we consider the following problem.

**Problem 2.** What is the complexity of  $k$ -IMP 2-COL restricted to planar graphs of maximum degree  $2k+2$ ?

We show in Section 2 that it is  $\mathcal{NP}$ -complete when  $k \in \{1, 2\}$ . Note that for  $k=1$ , it settles Cowen *et al.* [8] question. However, we conjecture that if  $k$  is sufficiently large then  $k$ -IMP 2-COL can be polynomially decided, as the answer is always affirmative.

**Conjecture 1.** *There exists an integer  $k_0 \geq 3$  such that for any  $k \geq k_0$ , any planar graph with maximum degree at most  $2k+2$  is  $k$ -improperly 2-colourable.*

We end the introduction with two definitions. Given an undirected graph  $G$ , an *orientation* of  $G$  is any directed graph obtained from  $G$  by assigning a unique direction to each edge. Then, a vertex  $u$  is *dominated* by a vertex  $v$  if there is an edge between  $v$  and  $u$  directed from  $v$  to  $u$ .

## 1 Complexity of $k$ -IMP $\ell$ -COL for graphs with maximum degree $(k+1)\ell$

In this section, we study the complexity of  $k$ -IMP  $\ell$ -COL restricted to graphs with maximum degree  $(k+1)\ell$ . The main result is the following theorem establishing the  $\mathcal{NP}$ -completeness of the problem when  $k \geq 1$  and  $\ell + \sqrt{\ell} \leq 2k+3$ .

**Theorem 2.** *Fix a positive integer  $k$ , and an integer  $\ell \geq 3$  such that  $\ell + \sqrt{\ell} \leq 2k+3$ . The following problem is  $\mathcal{NP}$ -complete:*

INSTANCE: *a graph  $G$  with maximum degree at most  $(k+1)\ell$ .*

QUESTION: *is  $G$   $k$ -improperly  $\ell$ -colourable?*

**Remark 3.** The maximum value of  $\ell$  is approximately  $2k+4-\sqrt{2k+2}$ .

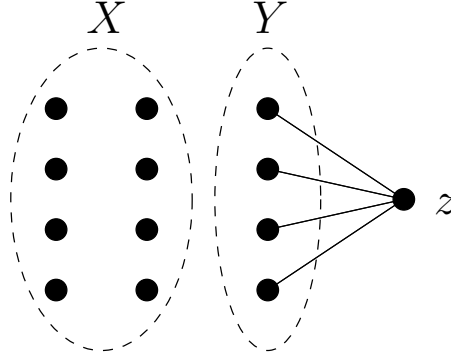


Figure 1: The complement of the graph  $H(3, 3)$ .

To prove Theorem 2, we need some preliminaries. Let  $k$  and  $\ell$  be two positive integers, and let  $H(k, \ell)$  be the graph with vertex set  $X \cup Y \cup \{z\}$  where  $|X| = (k + 1)(\ell - 1)$  and  $|Y| = (k + 1)$  such that  $xy$  is an edge unless  $x = z$  and  $y \in Y$  (see Figure 1).

**Proposition 4.** *The graph  $H(k, \ell)$  is  $k$ -improperly  $\ell$ -colourable, and in any  $k$ -improper  $\ell$ -colouring of  $H(k, \ell)$  the vertices of  $Y \cup \{z\}$  are coloured the same.*

*Proof.* Since  $H(k, \ell)$  has  $(k + 1)\ell + 1$  vertices, at least one colour class must contain at least  $k + 2$  vertices. Observe that a vertex of  $X$  must be in a colour class containing at most  $k + 1$  vertices since it is connected to every other vertex. Hence, the colour class with  $k + 2$  vertices is  $Y \cup \{z\}$ .  $\square$

**Lemma 5.** *Let  $G$  be a graph with maximum degree at most  $2k + 2$ . Then  $G$  has an orientation  $D$  such that every vertex has indegree and outdegree at most  $k + 1$ .*

*Proof.* Since every graph with maximum degree at most  $2k + 2$  is a subgraph of a  $(2k + 2)$ -regular graph, it suffices to prove the assertion for  $(2k + 2)$ -regular graphs. Let  $G'$  be such a graph, then it admits an Eulerian cycle  $C$ . Let  $D$  be the orientation of  $G'$  such that  $(u, v)$  is an arc if and only if  $u$  precedes  $v$  in  $C$ . Then  $D$  has indegree and outdegree at most  $k + 1$ .  $\square$

**Proof of Theorem 2.** Reduction to the following problem:

INSTANCE: a graph  $G$  with maximum degree at most  $2k + 2 \geq \ell + 1 \geq 4$ .

QUESTION: is  $G$   $\ell$ -colourable?

Thanks to the choice of  $\ell$ , this problem is  $\mathcal{NP}$ -complete by the result of Emden-Weinert *et al.* [10] cited in the introduction.

Let  $G = (V, E)$  be a graph of maximum degree at most  $2k + 2$ , and let  $D$  be an orientation of  $G$  with in- and outdegree at most  $k + 1$ . Such an orientation exists by Lemma 5.

Let  $G'$  be the graph constructed as follows: replace each vertex  $v$  of  $G$  by a copy  $H(v)$  of  $H(k, \ell)$ ; if  $v$  dominates  $u$  in  $D$  then connect  $z(v)$  to an element of  $Y(u)$ , in such a way

that every vertex of  $Y(u)$  is connected to a single vertex not in  $H(u)$ . The maximum degree of  $G'$  is  $(k+1)\ell$ .

Let us now prove that  $G$  is  $\ell$ -colourable if and only if  $G'$  is  $k$ -improperly  $\ell$ -colourable. If  $G$  admits an  $\ell$ -colouring  $c$ , then for any vertex  $v$  we assign the colour  $c(v)$  to the vertices of  $Y(v) \cup \{z(v)\}$  and the  $\ell-1$  other colours to  $\ell-1$  disjoint sets of  $k+1$  vertices of  $X(v)$ . This yields a  $k$ -improper  $\ell$ -colouring of  $G'$ .

Conversely, suppose that  $G'$  admits a  $k$ -improper  $\ell$ -colouring  $c'$ . Let  $c$  be defined by  $c(v) := c'(z(v))$ . We prove now that  $c$  is a proper  $\ell$ -colouring of  $G$ : let  $u$  and  $v$  be two neighbours. Without loss of generality,  $v$  is the predecessor of  $u$  in  $D$ . Thus, the vertex  $z(v)$  is connected to an element  $y(u)$  of  $Y(u)$ . Note that  $c'(z(v)) \neq c'(z(u))$ , otherwise by Proposition 4, all the vertices of  $Y(u) \cup Y(v) \cup \{z(u), z(v)\}$  are coloured the same. Then  $y(u)$  would have degree  $k+1$  in this set which is impossible. This concludes the proof.  $\square$

We now extend this result to the case when  $k=1$  and  $\ell=4$ .

**Theorem 6.** *The following problem is  $\mathcal{NP}$ -complete:*

INSTANCE: *a graph  $G$  with maximum degree at most 8.*

QUESTION: *is  $G$  1-improperly 4-colourable?*

Let  $B$  be the graph with vertex set  $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ , and  $xy$  is an edge except if there exists  $i \in \{1, 2, 3\}$  such that  $\{x, y\} = \{a_i, b_i\}$ .

Let  $A$  be the graph with vertex set  $\{x_1, x_2, y_1, y_2\}$  and with the two edges  $x_1x_2$  and  $y_1y_2$ . For  $i \in \{2, 3, 4\}$ , let  $J_i$  be the union of a copy  $A_i$  of  $A$  and a copy  $B_i$  of  $B$ , to which we add every edge  $xy$  such that  $(x, y) \in A_i \times B_i$ . Let  $A'$  be the graph obtained from  $A$  by removing the edge  $y_1y_2$ . We define  $J_1$  to be the union of a copy  $A_1$  of  $A'$  and a copy  $B_1$  of  $B$ . We let  $J'_1$  be a copy of  $J_1$  (with  $A'_1$  and  $B'_1$  defined analogously).

Let  $H := J'_1 \cup \bigcup_{i=1}^4 J_i$ , to which we add the following edges (see Figure 2):

$$\begin{array}{lll} y_1^{J_1} y_1^{J_2}, & y_1^{J_1} y_1^{J_3}, & y_2^{J_1} y_1^{J_4} \\ y_1^{J'_1} y_2^{J_2}, & y_1^{J'_1} y_2^{J_3}, & y_2^{J'_1} y_2^{J_4} \\ x_1^{J_2} x_2^{J_4}, & x_2^{J_2} x_1^{J_3}, & x_2^{J_3} x_1^{J_4} \end{array}$$

**Proposition 7.** *The graph  $H$  is 1-improperly 4-colourable, and for any 1-improper 4-colouring of  $H$ , the sets  $A_i$ , for  $i \in \{1, 2, 3, 4\}$ , and  $A_1 \cup A'_1$  are monochromatic.*

*Proof.* Consider a 1-improper 4-colouring of  $H$ . For every  $i \in \{1, 2, 3, 4\}$ , the colour of each vertex not belonging to  $A_i$  is assigned at most twice. Therefore, all the vertices of  $A_i$  must be coloured the same. The same holds also for  $A'_1$ . Moreover, for every  $j \in \{2, 3, 4\}$ , the colour of the vertices of  $A'_1$  and of  $A_j$  must be different from the colour of the vertices of  $A_i$  for every  $i \in \{1, 2, 3, 4\} \setminus \{j\}$ . Hence,  $A_1$  and  $A'_1$  are coloured the same.  $\square$

**Proof of Theorem 6.** Reduction to the following problem, which is  $\mathcal{NP}$ -complete [10]:

INSTANCE: a graph  $G$  with degree at most 6.

QUESTION: is  $G$  4-colourable?

Let  $G = (V, E)$  be a graph of maximum degree 6. By Lemma 5, let  $D$  be an orientation of  $G$  with in- and outdegree at most  $k + 1$ .

Let  $G'$  be the graph obtained by replacing each vertex  $v$  of  $G$  by a copy  $H(v)$  of  $H$ ; we set  $X(v) := \{x_1^{J_1}(v), x_2^{J_2}(v), x_1^{J'_1}(v)\}$  and  $Y(v) := \{y_2^{J_1}(v), y_2^{J'_1}(v), x_2^{J'_1}(v)\}$ ; if  $v$  dominates  $u$  in  $D$ , then we connect an element of  $Y(v)$  to an element of  $X(u)$  in such a way that every vertex of  $X(v) \cup Y(v)$  is connected to a single vertex not in  $H(v)$ , see Figure 2.

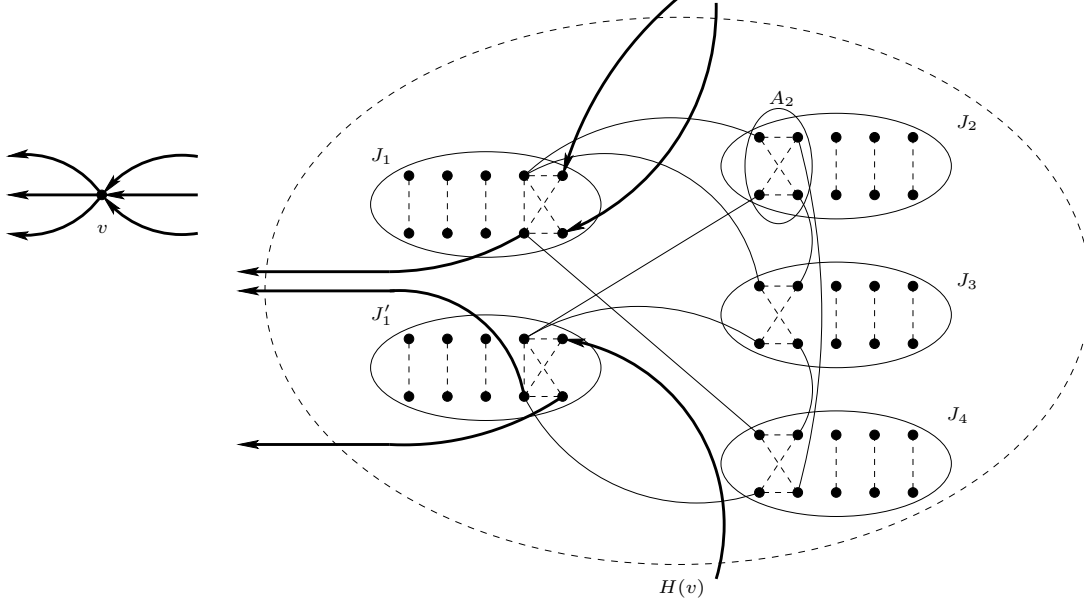


Figure 2: Replacing a vertex  $v$  of  $G$  by a copy  $H(v)$  of  $H$ . For each subgraph  $J_i$  or  $J'_1$ , dotted lines indicate missing edges.

The maximum degree of  $G'$  is 8. Moreover  $G'$  is 1-improper 4-colourable if and only if  $G$  is 4-colourable. Indeed, consider any proper 4-colouring of  $G$ . For every vertex  $v \in V$ , assign to every vertex of  $A_1(v) \cup A'_1(v)$  the colour of  $v$ . This partial colouring of  $G'$  can be extended to a 1-improper 4-colouring of  $G'$ . Conversely, if  $C'$  is a 1-improper 4-colouring of  $G'$ , then by Proposition 7, for each  $v \in V$  the vertices of  $A_1(v)$  are monochromatic. For all  $v \in V$ , we define  $C(v)$  to be the colour assigned to every vertex of  $A_1(v)$ . Then  $C$  is a proper 4-colouring of  $G$ : consider an edge  $uv$  of  $G$ , and say it is oriented from  $u$  to  $v$  in  $D$ . By Proposition 7, the vertex of  $X(u)$  to which the corresponding edge of  $G'$  is linked has impropriety 1 in  $H(u)$ . Thus, the vertex of  $Y(v)$  to which it is linked is coloured differently. Consequently, by the definition of  $C$ , we deduce that  $C(u) \neq C(v)$ , as desired.  $\square$

## 2 Planar graphs

In this section we show that 1-IMP 2-COL and 2-IMP 2-COL are  $\mathcal{NP}$ -complete when restricted to planar graphs of bounded maximum degree.

**Theorem 8.** *The following problem is  $\mathcal{NP}$ -complete:*

INSTANCE: *a planar graph  $G$  with maximum degree 4.*

QUESTION: *is there a 1-improper 2-colouring of  $G$  ?*

*Proof.* This result is proved by slightly modifying the proof of Cowen *et al.* [8]. The only change is in the crossing gadget. The crossing gadget of Cowen *et al.* [8] has maximum degree 5, and they asked whether one with maximum degree 4 exists. We shall exhibit such a crossing gadget. We make the whole proof here for completeness.

The reduction is from 3-SAT. Let  $\Phi$  be a 3-CNF. We shall construct, in polynomial time, a planar graph  $G_\Phi$  of maximum degree 4 such that  $\Phi$  is satisfiable if and only if  $G_\Phi$  is 1-improperly 2-colourable. We use several gadgets with useful properties.

An *xy-regulator* is depicted in Figure 3. There is a unique 1-improper 2-colouring of this graph, in which  $x$ ,  $u_1$ ,  $u_2$ , and  $y$  form a colour-class and all have impropriety zero. Note also that both  $x$  and  $y$  have degree 1 within an *xy-regulator*. An *xy-inversor*, depicted in Figure 4, is a  $K_{2,3}$ , with  $x$  and  $y$  being any two vertices not in the same part of the bipartition. There is a unique 1-improper 2-colouring of this graph, in which  $x$  and  $y$  receive different colours, and both have impropriety zero. Note that one out of  $x$  and  $y$  has degree 2 in the inversor while the other has degree 3. The variable-gadget that represents the literals of a particular variable is constructed from the two aforementioned graphs as shown in Figure 5. We put one variable-gadget for each variable, with as many literals as needed. Each of the literals will be linked to exactly one clause.

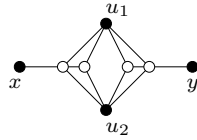


Figure 3: A regulator and its unique 1-improper 2-colouring.

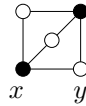


Figure 4: An inversor and its unique 1-improper 2-colouring.

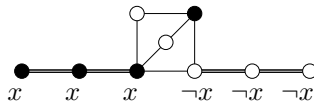


Figure 5: The vertex gadget. A double edge stands for a regulator.



Let  $C_1, C_2, \dots, C_m$  be the clauses of  $\Phi$ . For each clause  $C_i$ , we put a copy  $G'_i$  of the graph  $G'$ . It is left to the reader to check out that the graph  $G'$ , depicted in Figure 6, has the following property: in any 1-improper 2-colouring,  $z$  is coloured 1 if and only if at least one of  $p, q, r$  is. Then we add the edges  $z_i z_{i+1}$ , for  $i \in \{1, 2, \dots, m-1\}$ .

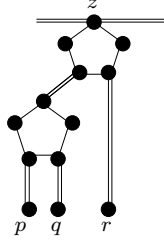


Figure 6: The clause gadget. A double edge stands for a regulator.

The obtained graph has maximum degree 4, and the vertex-gadgets and the clauses can be arranged so that the only edges that can cross are the ones joining vertex-gadgets to clauses. We uncross two crossing edges by using the crossing gadget  $CG$  depicted in Figure 7. The graph  $CG$  has maximum degree 4, and fulfils the following properties:

- (i) the graph  $CG$  is 1-improperly 2-colourable but not 1-improperly 1-colourable;
- (ii) in any 1-improper 2-colouring of  $CG$ , for  $i \in \{1, 2\}$ , the vertices  $a_i$  and  $b_i$  are coloured the same; and
- (iii) there exist two 1-improper 2-colourings  $c_1$  and  $c_2$  of  $CG$  such that  $c_1(a_1) = c_1(b_1) = c_1(a_2) = c_1(b_2)$  and  $c_2(a_1) = c_2(b_1) \neq c_2(a_2) = c_2(b_2)$ .

Properties (i) and (iii) can be directly checked. For property (ii), suppose that  $c$  is a 1-improper 2-colouring of  $CG$ . First, observe that, necessarily, three vertices among  $o_1, o_2, o_3$ , and  $o_4$  are coloured the same, because the vertex  $o_0$  is linked to all of them. Thus, the vertices  $o_2$  and  $o_3$  must be coloured differently. If  $c(a_1) \neq c(b_1)$ , then  $c(o_4) \neq c(o_1)$  and by the preceding observation, the vertex  $o_0$  has two neighbours of each colour, a contradiction. Now, suppose that  $c(a_2) \neq c(b_2)$  and so  $c(o_3) \neq c(o_7)$ . By the construction,  $c(o_7) = c(o_6) \neq c(o_5)$ , because  $o_6$  and  $o_7$  are joined by a regulator and  $o_5$  and  $o_6$  are joined by an invisor. Hence  $c(o_3) = c(o_5)$ , and so  $c(o_4) \neq c(o_3)$ . Therefore,  $c(o_4) = c(o_2) = c(o_1)$ . This is not possible since  $c(o_4) = c(o_7)$ , so the vertex  $o_1$  would have impropriety 2.

It remains to show that  $\Phi$  is satisfiable if and only if the obtained graph  $G_\Phi$ , which is planar and of maximum degree 4, is 1-improperly 2-colourable. Consider a 1-improper 2-colouring of  $G_\Phi$ . Without loss of generality, say that each clause — vertices  $z$  of the clause gadgets — is coloured 1. Then, at least one of the input vertices of each clause is coloured 1. Therefore, associating 1 with TRUE and 2 with FALSE yields a truth assignment for  $\Phi$ . Conversely, starting from a truth assignment of  $\Phi$ , one can derive a 1-improper 2-colouring of  $G_\Phi$  as follows. Vertices corresponding to literals are coloured 1 if the corresponding

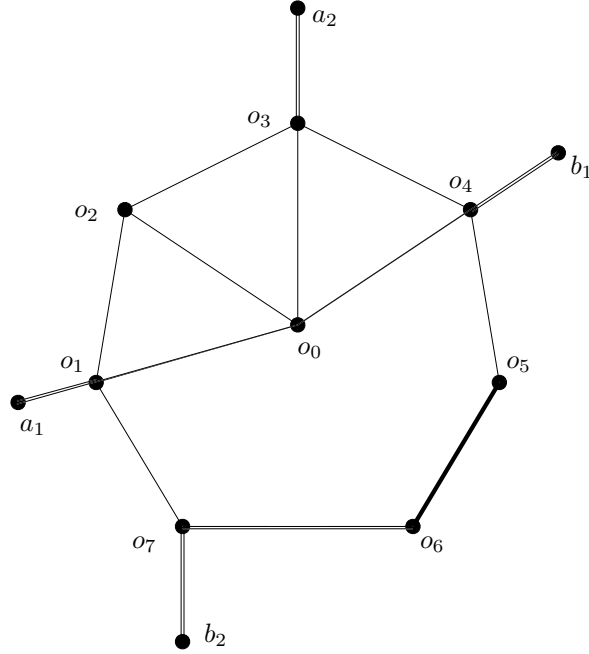


Figure 7: The crossing gadget  $CG$ . A double edge stands for a regulator, and the bold edge stands for an inversor.

literal is TRUE, and 2 otherwise. Thanks to the properties of the gadgets, such a partial colouring can be extended to a 1-improper 2-colouring of  $G_\Phi$ .  $\square$

**Theorem 9.** *The following problem is  $\mathcal{NP}$ -complete:*

INSTANCE: *a planar graph  $G$  of maximum degree 6.*

QUESTION: *is there a 2-improper 2-colouring of  $G$ ?*

*Proof.* We shall reduce the problem to 1-improper 2-colouring of planar graphs with maximum degree 4. Let  $G$  be a planar graph of maximum degree 4: we construct, in polynomial time, a planar graph  $\hat{G}$  of maximum degree 6 that is 2-improperly 2-colourable if and only if  $G$  is 1-improperly 2-colourable. The graph  $H$  showed in Figure 8 fulfils the following properties:

- (i) it is planar and has maximum degree 6;
- (ii) in any 2-improper 2-colouring of  $H$ , the vertex  $v$  must have impropriety at least 1; and
- (iii) there exists a 2-improper 2-colouring of  $H$  in which the vertex  $v$  has impropriety exactly 1.

Property (i) can be directly checked. To prove (ii), it is sufficient to show that no 2-improper 2-colouring of  $H$  such that  $a$  and  $b$  are coloured the same exists. So, suppose that  $c(a) = c(b) = 1$ .

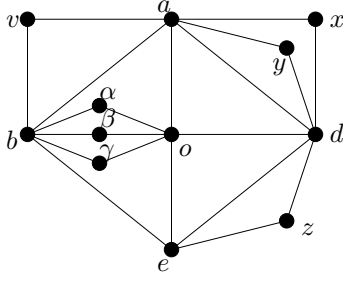


Figure 8: The graph  $H$ .

If  $c(o) = 1$ , then  $d$ ,  $y$ , and  $x$  must be coloured 2, so  $e$  and  $z$  both receive colour 1. Therefore  $e$  has impropriety 3, because of  $b$ ,  $o$ , and  $z$ , a contradiction.

If  $c(o) = 2$ , then the three vertices  $\alpha, \beta, \gamma$  cannot be coloured the same, otherwise  $b$  or  $o$  would have impropriety at least 3. Moreover, since  $c(b) = c(a) = 1$ , exactly two vertices among  $\alpha, \beta, \gamma$  are coloured with colour 2. Hence, the vertices  $b$  and  $o$  both have impropriety 2 in the subgraph of  $G$  induced by the vertices  $a, b, o, \alpha, \beta, \gamma$ . But the vertex  $e$  does not belong to this subgraph, and is linked to both  $b$  and  $o$ , a contradiction. This proves **(ii)**.

Assigning 1 to  $\{v, a, d, e, \alpha, \beta\}$  and 2 to  $\{b, o, x, y, z, \gamma\}$ , we obtain the colouring of **(iii)**.

To construct the graph  $\hat{G}$ , put a copy  $H(x)$  of  $H$  for each vertex  $x \in V(G)$ . Then, for each edge  $xy \in E(G)$ , we put an edge between the vertex  $v$  of  $H(x)$  and the vertex  $v$  of  $H(y)$ . Note that  $H$  has maximum degree 6 and  $v$  has degree 2 in  $H$ , so, as  $G$  has maximum degree 4, the graph  $\hat{G}$  has maximum degree 6. Furthermore, the graph  $\hat{G}$  is planar.

Now let  $c$  be a 1-improper 2-colouring of  $G$ . For any  $x \in V(G)$ , assign the colour  $c(x)$  to the vertex  $v$  of  $H(x)$ , and then extend the colouring to each copy of  $H$  by property **(iii)**.

If  $c$  is a 2-improper 2-colouring of  $\hat{G}$ , then for each  $x \in V(G)$ , assign to  $x$  the colour of the vertex  $v$  of  $H(x)$ . The obtained 2-colouring of  $G$  is 1-improper because of property **(ii)** of  $H$ .  $\square$

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